

The von Neumann entropy asymptotics in multidimensional fermionic systems

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We study the von Neumann entropy asymptotics of pure translation-invariant quasi-free states of d -dimensional fermionic systems. It is shown that the entropic area law is violated by all these states: apart from the trivial cases, the entropy of a cubic subsystem with edge length L cannot grow slower than $L^{d-1} \ln L$. As for the upper bound of the entropy asymptotics, the zero-entropy-density property of these pure states is the only limit: it is proven that arbitrary fast sub- L^d entropy growth is achievable.

I. Introduction

The structure of reduced density matrices belonging to subsystems of lattice models has been the focus of several recent studies.^{1–21} Especially, the von Neumann entropy of the reduced density matrices, defined as $S(\rho) := -\text{Tr} \rho \ln \rho$, and its growth with the size of the subsystem have attracted much attention. In the context of condensed matter physics, a non-saturating entropy asymptotics of the ground states of one-dimensional lattice models was found to be an indicator of quantum criticality,¹ moreover, this asymptotics was connected by conformal field theoretical methods to the central charge of the theory.² For multidimensional lattice systems the situation proved to be more complicated, namely, the von Neumann entropy asymptotics of ground states was found to depend on the statistics of the fields defining the model. For free bosonic Hamiltonians with nearest neighbor hopping terms the ground-state entropy was shown to grow with the area of the surface of the subsystem (regardless of the criticality of the system),⁵ while for free fermionic Hamiltonians with similar hopping terms a logarithmic correction to the area-law was found.^{6–9} The statistics of the fields plays also an important

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role in the interpretation of the von Neumann entropy as a measure of entanglement in quantum information theory. Such an interpretation does not work for fermionic systems,^{22,23} but in the case of bosonic and spin systems the von Neumann entropy of a subsystem is a natural measure of entanglement if the whole system is in a pure state.²⁴ Another motivation comes from data compression and DMRG theory: it is believed that for translation-invariant states the von Neumann entropy of the reduced density matrix is related to the dimension of the "essential subspace" of the restricted state.^{16,17} Finally, let us mention two mathematical conjectures about the entropy asymptotics. A long-standing conjecture, called the zero-entropy-conjecture (see e.g. Refs. [3,4]), states that the entropy density $\lim_{L \rightarrow \infty} S_L/L^d$ of pure translation-invariant states of lattice spin and fermionic systems is zero. Where S_L denotes the von Neumann entropy of the d -dimensional cubic subsystem with edge length L . Recently, it has also been conjectured that the type of the von Neumann algebra obtained as the weak closure of the C^* -algebra belonging to the left (or right) half of a spin chain in a given state depends also on the von Neumann entropy asymptotics of the state.¹⁸

For the above (and also many other) reasons the von Neumann entropy asymptotics of several one- and multidimensional states has been studied. The most investigated states are pure translation-invariant quasi-free states of bosonic and fermionic systems. As mentioned previously, the entropy asymptotics in these two cases are quite different. In the bosonic case an entropic area law was shown to hold,⁵ while for certain gauge-invariant fermionic quasi-free states the entropy asymptotics of cubic subsystems was shown to be $L^{d-1} \ln L$ (where L is the edge length of the cubic subsystem, and d is the dimensionality of the system), but it was also hinted (see e.g. Refs. [6,7]) that the conditions made on the Fermi surface might exclude somewhat exotic, but physically relevant states. In this article we prove that any nontrivial (gauge-) and translation-invariant quasi-free state gives at least an $L^{d-1} \ln L$ entropy asymptotics, but the entropy asymptotics can also be much faster than $L^{d-1} \ln L$. Actually, we prove, that the zero-entropy-density conjecture is sharp in the sense that for any function F_L that has a sub- L^d asymptotics, one can find a state which has a faster asymptotics than F_L .

II. Translation-invariant quasi-free states

In this section we shortly recall a few facts about quasi-free states in order to be self-contained. For much more detailed treatments see Refs. [25–27], where also the proofs of the statements mentioned in this section can be found.

The observable algebra \mathcal{A}_d of a fermionic system on the d -dimensional cubic lattice \mathbb{Z}^d is the CAR algebra corresponding to the Hilbert space $\ell^2(\mathbb{Z}^d)$, i.e., it is the C^* -algebra generated by $\mathbb{1}$ and $\{c_{\mathbf{k}} : \mathbf{k} = (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d\}$, satisfying the

canonical anticommutation relations:

$$\begin{aligned} c_{\mathbf{k}}c_{\mathbf{k}'} + c_{\mathbf{k}'}c_{\mathbf{k}} &= 0, \\ c_{\mathbf{k}}^*c_{\mathbf{k}'} + c_{\mathbf{k}'}^*c_{\mathbf{k}} &= \delta_{\mathbf{k},\mathbf{k}'} \mathbb{1}. \end{aligned}$$

The translation automorphisms $\alpha_i : \mathcal{A}_d \rightarrow \mathcal{A}_d$ corresponding to the d different canonical unit translations of \mathbb{Z}^d are given by

$$\alpha_1(c_{\mathbf{k}}) = c_{\mathbf{k}+(1,0,0,\dots,0,0)}, \quad \alpha_2(c_{\mathbf{k}}) = c_{\mathbf{k}+(0,1,0,\dots,0,0)}, \quad \dots \quad \alpha_d(c_{\mathbf{k}}) = c_{\mathbf{k}+(0,0,0,\dots,0,1)}.$$

A state ω on \mathcal{A}_d is called translation-invariant if $\omega \circ \alpha_i = \omega$ for every α_i .

Let Q be a positive bounded operator on $\ell^2(\mathbb{Z}^d)$ for which $0 \leq Q \leq \mathbb{1}$ holds. The gauge-invariant quasi-free state ω_Q belonging to this operator is defined by the rule:

$$\omega_Q(c_{\mathbf{k}_1}^* \dots c_{\mathbf{k}_n}^* c_{\mathbf{l}_m} \dots c_{\mathbf{l}_1}) = \delta_{m,n} \det \left([Q_{\mathbf{k}_i, \mathbf{l}_j}]_{i,j=1}^n \right),$$

where $Q_{\mathbf{k},\mathbf{l}}$ denotes the matrix elements $\langle \psi_{\mathbf{k}}, Q \psi_{\mathbf{l}} \rangle$. Here $\psi_{\mathbf{k}}$ is the characteristic function of the lattice point \mathbf{k} (i.e. $\psi_{\mathbf{k}}(\mathbf{k}') = \delta_{\mathbf{k},\mathbf{k}'}$). Q is called the symbol of the quasi-free state ω_Q .

Translation-invariant quasi-free states are characterized by certain integrable functions on the torus $\mathbb{T}^d = \times_{i=1}^d S^1$. Let us parametrize the d -dimensional torus \mathbb{T}^d by $[-\pi, \pi)^d$. A gauge-invariant quasi-free state ω_Q is translation-invariant if and only if there exists an integrable function $q : [-\pi, \pi)^d \rightarrow [0, 1)$ such that

$$Q_{\mathbf{k},\mathbf{l}} = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} da_1 \dots \int_{-\pi}^{\pi} da_d q(a_1, \dots, a_d) e^{-i[(l_1 - k_1)a_1 + \dots + (l_d - k_d)a_d]}.$$

Furthermore, this state is pure if and only if q is (almost everywhere) equal to the characteristic function of a measurable set $\mathbb{M} \subset \mathbb{T}^d$. Such a quasi-free pure state belonging to the measurable set \mathbb{M} will be denoted by $\omega_{\mathbb{M}}$. \mathbb{M} is called the "Fermi sea", while the boundary of the interior points of \mathbb{M} is called the "Fermi surface" of the state $\omega_{\mathbb{M}}$.

III. Entropy asymptotics of quasi-free states

Let ρ_L denote the density matrix obtained by restricting the quasi-free state ω_Q to the subalgebra corresponding to the lattice points $\{1, 2, \dots, L\}^d \subset \mathbb{Z}^d$. The von Neumann entropy of the restricted state, $S_L := -\text{Tr } \rho_L \ln \rho_L$, can be expressed in terms of Q as:²⁵

$$S_L = -\text{Tr } (Q_L \ln Q_L + (\mathbb{1} - Q_L) \ln(\mathbb{1} - Q_L)),$$

where Q_L is the restriction of Q to the $L^d \times L^d$ submatrix corresponding to the points $\{1, 2, \dots, L\}^d \subset \mathbb{Z}^d$. The inequality $-x \ln x - (1-x) \ln(1-x) \geq x(1-x)$, which holds for $0 \leq x \leq 1$, implies that

$$S_L \geq \text{Tr } Q_L(\mathbb{1} - Q_L).$$

In the case of a pure (gauge- and) translation-invariant quasi-free state $\omega_{\mathbb{M}}$ the above lower bound can be rewritten, as shown in Refs. [3,6], in the form:

$$S_L \geq \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} da_1 \dots \int_{-\pi}^{\pi} da_d \prod_{i=1}^d k_L(a_i) \Lambda_{\mathbb{M}}(a_1, \dots, a_d). \quad (1)$$

The definitions of k_L and $\Lambda_{\mathbb{M}}$ are the following:

$$k_L(a) = \frac{\sin^2 La/2}{\sin^2 a/2}, \quad \text{and} \quad \Lambda_{\mathbb{M}}(\mathbf{a}) = |\mathbb{M} \setminus \mathbb{M} + \mathbf{a}|,$$

where $|\cdot|$ denotes the Lebesgue measure, and for any $\mathbf{a} = (a_1, a_2, \dots, a_d) \in \mathbb{R}^d$ -vector $\mathbb{M} + \mathbf{a}$ is the image of \mathbb{M} after a translation of the points of the torus $\mathbb{T}^d = \times_{i=1}^d S^1$ defined by rotating the first S^1 by a_1 , the second S^1 factor by a_2 , and so on. Hence the vectors \mathbf{a} and $\mathbf{a} + (2\pi n_1, 2\pi n_2, \dots, 2\pi n_d)$, where $n_i \in \mathbb{Z}$, act on the torus in the same way. The lower bound (1), which was first developed by Fannes, Haegeman, and Mosonyi [3], will be the starting point of both of our theorems.

A. Lower bound on the entropy asymptotics

One can immediately observe that if the symbol of a pure quasi-free is 0 or $\mathbb{1}$ (i.e., if $|\mathbb{M}|=0$ or $|\mathbb{M}| = |\mathbb{T}^d|$), then $S_L = 0$. We show in this section that all the other pure translation-invariant quasi-free states of d -dimensional fermionic systems have at least an $L^{d-1} \ln L$ entropy asymptotics.

Theorem 1. *Let $\omega_{\mathbb{M}}$ be a pure (gauge- and) translation-invariant quasi-free state for which $0 < |\mathbb{M}| < |\mathbb{T}^d|$. The entropy growth S_L of $\omega_{\mathbb{M}}$ is bounded from below by $cL^{d-1} \ln L$ for some $c > 0$ (which depends on \mathbb{M}).*

Proof. The proof is divided into four steps. First we investigate general properties of $\Lambda_{\mathbb{M}}$. Then putting everything together, we obtain a lower bound for $\Lambda_{\mathbb{M}}$ by the aid of which the proof can be easily completed in the last step.

1. Continuity and subadditivity of $\Lambda_{\mathbb{M}}$

The continuity of $\Lambda_{\mathbb{M}}$ can be proven from Stone's theorem. According to this theorem the representation of the translations in $L^2(\mathbb{T}^d)$ given by $(U_{\mathbf{a}}\psi)(\mathbf{x}) = \psi(\mathbf{x} + \mathbf{a})$ is continuous in the strong topology, hence in the weak topology as well. Let $\chi_{\mathbb{M}}$ be the characteristic function of \mathbb{M} . The difference $\Lambda_{\mathbb{M}}(\mathbf{b}) - \Lambda_{\mathbb{M}}(\mathbf{a})$ can be written as

$$\begin{aligned} \Lambda_{\mathbb{M}}(\mathbf{b}) - \Lambda_{\mathbb{M}}(\mathbf{a}) &= \int_{\mathbb{T}^d} \chi_{\mathbb{M}}(\mathbf{x})(1 - \chi_{\mathbb{M}}(\mathbf{x} + \mathbf{b})) - \int_{\mathbb{T}^d} \chi_{\mathbb{M}}(\mathbf{x})(1 - \chi_{\mathbb{M}}(\mathbf{x} + \mathbf{a})) = \\ &= \int_{\mathbb{T}^d} \chi_{\mathbb{M}}(\mathbf{x})(\chi_{\mathbb{M}}(\mathbf{x} + \mathbf{a}) - \chi_{\mathbb{M}}(\mathbf{x} + \mathbf{b})) = \langle \chi_{\mathbb{M}}, (U_{\mathbf{a}} - U_{\mathbf{b}})\chi_{\mathbb{M}} \rangle. \end{aligned}$$

Weak continuity of $U_{\mathbf{a}}$ implies that this expression goes to zero as \mathbf{a} goes to \mathbf{b} , thus $\Lambda_{\mathbb{M}}$ is continuous.

Next, for any two translations \mathbf{a} and \mathbf{b} the following holds:

$$\mathbb{M} \setminus (\mathbb{M} + \mathbf{a} + \mathbf{b}) \subset [\mathbb{M} \setminus (\mathbb{M} + \mathbf{a})] \cup [(\mathbb{M} + \mathbf{a}) \setminus (\mathbb{M} + \mathbf{a} + \mathbf{b})].$$

By monotony and translational invariance of the Lebesgue measure we obtain the subadditivity property:

$$\Lambda_{\mathbb{M}}(\mathbf{a} + \mathbf{b}) \leq \Lambda_{\mathbb{M}}(\mathbf{a}) + \Lambda_{\mathbb{M}}(\mathbf{b}).$$

2. Irrelevant and relevant directions of $\Lambda_{\mathbb{M}}$

The subspace $\{\kappa \mathbf{a} : \kappa \in \mathbb{R}\}$ generated by a vector $\mathbf{a} \in \mathbb{R}^d$ is called an *irrelevant direction* (with respect to \mathbb{M}) if $\Lambda_{\mathbb{M}}(\kappa \mathbf{a}) = 0$ for all $\kappa \in \mathbb{R}$, otherwise it is called a *relevant direction*. Subadditivity of $\Lambda_{\mathbb{M}}$ implies that vectors generating irrelevant directions form a vector space:

$$\Lambda_{\mathbb{M}}(\kappa(\alpha \mathbf{a} + \beta \mathbf{b})) \leq \Lambda_{\mathbb{M}}(\kappa \alpha \mathbf{a}) + \Lambda_{\mathbb{M}}(\kappa \beta \mathbf{b}) = 0.$$

However, if \mathbf{a} and \mathbf{b} generate relevant directions, then a linear combination of \mathbf{a} and \mathbf{b} can generate either a relevant or an irrelevant direction.

It is easy to show that there exists at least one relevant direction of $\Lambda_{\mathbb{M}}$ if \mathbb{M} is nontrivial (i.e., if $0 < |\mathbb{M}| < |\mathbb{T}^d|$). If all directions were irrelevant, then by definition, \mathbb{M} would remain invariant (up to a zero measure set) under any translation. In this case one could define a translation-invariant measure μ on every Lebesgue measurable set \mathbb{L} by the formula $\mu(\mathbb{L}) := |\mathbb{M} \cap \mathbb{L}|$. According to Haar's theorem any translation-invariant measure on the torus is equal to the Lebesgue measure times a constant, i.e., $\mu(\mathbb{M}) = k|\mathbb{M}|$. If $k = 0$ then $|\mathbb{M}| = |\mathbb{M} \cap \mathbb{M}| = \mu(\mathbb{M}) = 0$, if $k > 0$, then $|\mathbb{T}^d| = \mu(\mathbb{T}^d)/k = |\mathbb{M} \cap \mathbb{T}^d|/k = |\mathbb{M} \cap \mathbb{M}|/k = \mu(\mathbb{M})/k = |\mathbb{M}|$.

Let \mathbf{e}_i denote the i th standard unit vector of \mathbb{R}^d , $\mathbf{e}_1 = (1, 0, 0, \dots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \dots, 0)$, etc. Vectors of the form $\kappa \mathbf{e}_i$ act on the torus $\mathbb{T}^d = \times_{i=1}^d S^1$ by rotating only the i th S^1 factor and leaving the other S^1 factors invariant. We call the one-parameter subspaces of the form $\{\kappa \mathbf{e}_i : \kappa \in \mathbb{R}\}$ *principal directions*. It follows from the previous discussion that all principal directions cannot be irrelevant (if \mathbb{M} is nontrivial). By a permutation of the S^1 factors, we can achieve that the first m ($m > 0$) standard unit vectors each generate a relevant direction, while the last $d - m$ generate irrelevant directions.

3. Lower bound for $\Lambda_{\mathbb{M}}$

First we show that for every fixed relevant direction there exists a linear lower bound. Let \mathbf{a} be a vector for which $\Lambda_{\mathbb{M}}(\mathbf{a}) > 0$. Continuity of $\Lambda_{\mathbb{M}}$ implies that there is an $\epsilon > 0$ and a $c > 0$ such that $\Lambda_{\mathbb{M}}(\nu \mathbf{a}) > c$ for any $1 - \epsilon \leq \nu \leq 1$. Let us

denote by $\lfloor x \rfloor$ the "lower integer part" of x ($x \geq \lfloor x \rfloor$). Now, $1 - \epsilon \leq \lfloor 1/\lambda \rfloor \lambda \leq 1$ holds if $0 < \lambda \leq \epsilon$. Using the subadditivity of $\Lambda_{\mathbb{M}}$, we obtain $c < \Lambda_{\mathbb{M}}(\lfloor 1/\lambda \rfloor \lambda \mathbf{a}) \leq \lfloor 1/\lambda \rfloor \Lambda_{\mathbb{M}}(\lambda \mathbf{a}) \leq \Lambda_{\mathbb{M}}(\lambda \mathbf{a})/\lambda$. Summarizing, for any \mathbf{a} that generates a relevant direction, there exist a $c > 0$ and an $\epsilon > 0$ so that

$$\Lambda_{\mathbb{M}}(\lambda \mathbf{a}) < c\lambda \quad \text{for } 0 < \lambda < \epsilon.$$

However, this is not enough for an estimate of the integrand in (1), which is our goal. Next we have to show that there exists a sufficiently large set of relevant directions. As we have mentioned in the previous part of the proof, we can assume that the first m standard basis vectors $\{\mathbf{e}_i\}_{i=1}^m$ each generate a relevant direction. This does not mean that a linear combination of them also generates a relevant direction, but we can circumvent this problem by finding an m -dimensional sub-region in which the positive linear combinations (positive cone) of vectors have this property.

If for every choice of signs $\{s_i\}_{i=1}^m$ a vector (not equal to zero or any of the first m standard basis vectors \mathbf{e}_i) is picked from the sets $\{\sum_{i=1}^m a_i(s_i \mathbf{e}_i) \mid a_i \geq 0\}$, then these vectors will linearly generate the whole \mathbb{R}^m vector space spanned by the standard basis vectors $\{\mathbf{e}_i\}_{i=1}^m$. Therefore there is a choice of signs $\{s_i\}_{i=1}^m$ such that any vector in the compact set $V := \{(s_1 a_1, s_2 a_2, \dots, s_m a_m, 0, \dots, 0) \mid a_i \geq 0, \sum_i a_i^2 = 1\}$ generates a relevant direction, since if such a choice did not exist, then all the directions (including the ones generated by the vectors $\{\mathbf{e}_i\}_{i=1}^m$) would be irrelevant, which contradicts our assumption.

For any relevant direction we have a linear lower bound for $\Lambda_{\mathbb{M}}$ if the translation is sufficiently small. Unfortunately, the prefactor and the validity region of the linear lower bound depend on the direction, so for a global lower bound of $\Lambda_{\mathbb{M}}$ we have to get rid of this direction dependence. For this purpose, first consider the following function defined on V :

$$s(\mathbf{v}) := \sup \{ c \mid \exists \epsilon > 0 \text{ so that } \Lambda_{\mathbb{M}}(\lambda \mathbf{v}) \geq c\lambda \text{ for any } \lambda \leq \epsilon \}.$$

We show that if $s_- = \inf_{\mathbf{v} \in V} s(\mathbf{v}) = 0$, then there would exist an irrelevant generator in V in contradiction to its definition, therefore s_- is positive. Since V is compact, if $s_- = 0$ then there is a sequence $\mathbf{v}_n \in V$, which is convergent, and $\lim_{n \rightarrow \infty} s(\mathbf{v}_n) = 0$. Let the limit of \mathbf{v}_n be \mathbf{v} . By subadditivity of $\Lambda_{\mathbb{M}}$ and the definition of the function s , for any positive integer k there is an index n_k so that $\Lambda_{\mathbb{M}}(\lambda \mathbf{v}_{n_k}) < \lambda/k$ for any λ . By continuity of $\Lambda_{\mathbb{M}}$,

$$\Lambda_{\mathbb{M}}(\lambda \mathbf{v}) = \lim_{k \rightarrow \infty} \Lambda_{\mathbb{M}}(\lambda \mathbf{v}_{n_k}) \leq \lim_{k \rightarrow \infty} \frac{\lambda}{k} = 0.$$

Let $0 < \sigma < s_-$. It is important that σ is strictly smaller than s_- . We define a function on V (whose σ -dependence is suppressed because σ is fixed from now on):

$$\epsilon(\mathbf{v}) := \sup \{ \epsilon \mid \Lambda_{\mathbb{M}}(\lambda \mathbf{v}) \geq \sigma \lambda \text{ if } \lambda \leq \epsilon \}.$$

We show that $\epsilon_- := \inf_{\mathbf{v} \in V} \epsilon(\mathbf{v}) > 0$. The argument is similar to the one we have just finished. Suppose the contrary. V is compact, so we have a convergent sequence \mathbf{v}_n , with limit \mathbf{v} , such that $\lim_{n \rightarrow \infty} \epsilon(\mathbf{v}_n) = 0$. Note that our choice $\sigma < s_-$ guarantees that ϵ is strictly positive on V . Continuity of $\Lambda_{\mathbb{M}}$ implies that $\Lambda_{\mathbb{M}}(\epsilon(\mathbf{u})\mathbf{u}) = \sigma\epsilon(\mathbf{u})$ for any $\mathbf{u} \in V$. Consequently,

$$\lim_{n \rightarrow \infty} \Lambda_{\mathbb{M}}\left(\left\lfloor \frac{\lambda}{\epsilon(\mathbf{v}_n)} \right\rfloor \epsilon(\mathbf{v}_n)\mathbf{v}_n\right) \leq \lim_{n \rightarrow \infty} \left\lfloor \frac{\lambda}{\epsilon(\mathbf{v}_n)} \right\rfloor \Lambda_{\mathbb{M}}(\epsilon(\mathbf{v}_n)\mathbf{v}_n) = \lim_{n \rightarrow \infty} \left\lfloor \frac{\lambda}{\epsilon(\mathbf{v}_n)} \right\rfloor \epsilon(\mathbf{v}_n)\sigma = \sigma\lambda$$

for any λ . But $\lim_{n \rightarrow \infty} \lfloor \lambda/\epsilon(\mathbf{v}_n) \rfloor \epsilon(\mathbf{v}_n)\mathbf{v}_n = \lambda\mathbf{v}$, and $\Lambda_{\mathbb{M}}(\lambda\mathbf{v}) > \sigma\lambda$ for some λ (the latter inequality is strict, this is the point where our choice $\sigma < s_-$ comes into play again), which contradicts the continuity of $\Lambda_{\mathbb{M}}$.

At last we arrived at the advertised lower bound for $\Lambda_{\mathbb{M}}$:

$$\Lambda_{\mathbb{M}}(\mathbf{v}) \geq \sigma \|\mathbf{v}\| \quad \text{if } \frac{\mathbf{v}}{\|\mathbf{v}\|} \in V, \text{ and } \|\mathbf{v}\| < \epsilon_-. \quad (2)$$

4. A lower bound for the entropy asymptotics

We can write the lower bound (1) as

$$\begin{aligned} S_L &\geq \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} da_1 \dots \int_{-\pi}^{\pi} da_m \prod_{i=1}^m k_L(a_i) \Lambda_{\mathbb{M}}(P_R \mathbf{a}) \left(\int_{-\pi}^{\pi} da_{m+1} \dots \int_{-\pi}^{\pi} da_d \prod_{i=d-m}^d k_L(a_i) \right) \\ &\geq \frac{L^{d-m}}{(2\pi)^m} \left| \int_0^{s_1 \epsilon_- / \sqrt{m}} da_1 \dots \int_0^{s_m \epsilon_- / \sqrt{m}} da_m \sigma \|P_R \mathbf{a}\| \right| \\ &\geq \frac{L^{d-m}}{(2\pi)^m} \sigma \int_0^{\epsilon_- / \sqrt{m}} da_1 \dots \int_0^{\epsilon_- / \sqrt{m}} da_m a_1 \prod_{i=1}^m k_L(a_i). \end{aligned}$$

In the first inequality we simply used the fact that the irrelevant translations alter \mathbb{M} only by a zero measure set, so in the argument of $\Lambda_{\mathbb{M}}$ the last $d - m$ components can be set to zero (P_R is the standard projection from $\mathbb{R}^d = \mathbb{R}^m \times \mathbb{R}^{d-m}$ to the subspace \mathbb{R}^m generated by the first m standard unit vectors), and the integrations over the irrelevant principal directions can be pulled out. Next, these integrals were performed, and the integration region was shrunk into a hypercube where the lower bound (2) can be applied. Finally, we replaced the Euclidean norm of $P_R \mathbf{a}$ with its first component.

Borrowing the inequalities

$$\int_0^{\delta} da k_L(a) \geq c_1 L, \quad \int_0^{\delta} da a k_L(a) \geq c_2 \ln L$$

from Ref. 3, which are valid in the case $L > 1$ for some $c_1 > 0$ and $c_2 > 0$, the proof is complete:

$$S_L \geq c L^{d-1} \ln L$$

with some constant $c > 0$. □

B. No sub- L^d upper bound for the entropy asymptotics

The result of the previous section was quite general: it holds for any translational-invariant pure quasi-free state. In Ref. [7] it was shown that for Fermi surfaces satisfying certain conditions the entropy asymptotics is in fact $c'L \ln L$. However, we show that the area law can be violated to a higher degree than logarithmic for general quasi-free states. In a sense, it can be broken to any extent permitted by the zero-entropy-density conjecture (which is in fact a theorem for the quasi-free states). The crux of the proof of this statement is the following observation:

Proposition. *Let h be a strictly monotonically increasing continuous function defined on the interval $[0, \epsilon]$ ($\epsilon > 0$). Furthermore, let $h(0) = 0$. There exists a set $M \subset S^1$ ($d = 1$) such that*

$$\Lambda_M(a) \geq h(a) \quad (3)$$

for sufficiently small a .

Proof. See Ref. [4]. □

Actually, the states constructed by the aid of M are not even so exotic; their Fermi sea is the union of (countably many) disjoint intervals. Now, we can repeat the proof of the theorem in Ref. [4] almost verbatim to show that it can be generalized to any spatial dimension d .

Theorem 2. *Let $F : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function that satisfies $\lim_{L \rightarrow \infty} F_L/L^d = 0$. There exists a pure quasi-free state such that $S_L \geq F_L$ for sufficiently large L .*

Proof. Let us define the function $f_L := F_L/L^d$. This satisfies $\lim_{L \rightarrow \infty} f_L/L = 0$. In Ref. [4] we argued that $h(x) := \frac{d}{dx}(xg(x))$ satisfies the conditions of the previous proposition if g is a suitably chosen positive function for which

$$\frac{2}{\pi^2} g\left(\frac{\pi}{L}\right) \geq \frac{f_L}{L}.$$

Therefore there exists a set $M \subset [-\pi, \pi)$ for which (3) holds with this particular h . Let $\mathbb{M} = \times_{i=1}^{d-1} [-\pi, \pi) \times M$. Then (1) simplifies to

$$S_L \geq \frac{1}{(2\pi)^d} \left(\int_{-\pi}^{\pi} da k_L(a) \right)^{d-1} \int_{-\pi}^{\pi} db k_L(b) \Lambda_M^{(1)}(b) = \frac{L^{d-1}}{2\pi} \int_{-\pi}^{\pi} db k_L(b) \Lambda_M^{(1)}(b).$$

Restricting the integration region and using (3), we obtain for sufficiently large L the final inequality:

$$S_L \geq \frac{L^{d-1}}{2\pi} \int_0^{\pi/L} db k_L(b) h(b) \geq \frac{L^{d-1}}{2\pi} \int_0^{\pi/L} db \frac{4L^2}{\pi^2} h(b) \geq \frac{2L^d}{\pi^2} g\left(\frac{\pi}{L}\right) = F_L$$

□

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